Context-free Pumping Lemma

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1 Context-free Pumping Lemma

The context-free pumping lemma is a powerful tool that helps you show that certain languages are not context-free. The statement of the context-free pumping lemma is as follows:

If L is a context-free language, then there exists an integer m such that given a word w in L, with $|w| \geq m$, w can be decomposed as $w = uvxyz$, with the following properties:

1. $|vxy| \leq m$
2. $|vy| > 0$, and
3. for each $i \geq 0$, $uv^ixy^iz \in L$.

To use the context-free pumping lemma to show that a given language L is not context-free, we proceed by assuming that L is context-free so that L satisfies the pumping lemma. Then we select an appropriate w in L and show that w cannot be decomposed as prescribed by the pumping lemma. This contradiction shows that our assumption must be wrong and hence L is not context-free.

We will use the language $L = \{ww : w \in \{a, b\}^*\}$. Thus L is the language of those even length words in $\{a, b\}^*$ where the first half of the string is identical to the second half.

We first give a direct argument to show that L is not regular. Then we use JFLAP to practice showing the same result.

2 Direct Proof

Assume that L is context-free. Then L satisfies the pumping lemma and hence there is an integer m such that the three properties above are satisfied. Select $w = a^mb^ma^mb^m$. Then $w \in L$ and $|w| \geq m$. By pumping lemma, $a^mb^ma^mb^m = w = uvxyz$. Now suppose the substring vxy is entirely contained in the first half of w. Then for uv^2xy^2z we get a b in the first position in the second half while the first letter in the first half is an a, so uv^2xy^2z cannot be in L. A similar argument applies if vxy is contained entirely in the second half of w. Hence vxy straddles the mid point of w. But in this case, for $i = 0$, the string uxz is of the form $a^mb^ra^sb^m$, where at least one of r and s is not m. Hence this string is not in L. This shows that L is not context-free.

3 Proof with JFLAP

JFLAP provides a proof by letting you play a game against the computer. The idea of the game is to win by providing a word w for which pumping lemma fails. Run JFLAP, select Context-free Pumping Lemma, select "You go first" and then select the language L as above from the list provided. Click on "Explain" button to see a description of why L is not context-free. Since you go first, JFLAP asks you to enter a number m. Enter 3. JFLAP selects a w. See Figure 1.

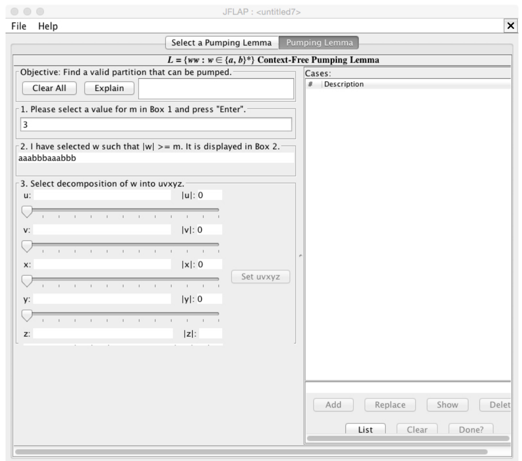

Figure 1: Step 1

JFLAP works be considering several cases for possible values of v and y in the decomposition $uvxyz$ of w. Click on the button "List" in the lower right to see all the cases for $m = 3$. See Figure 2.

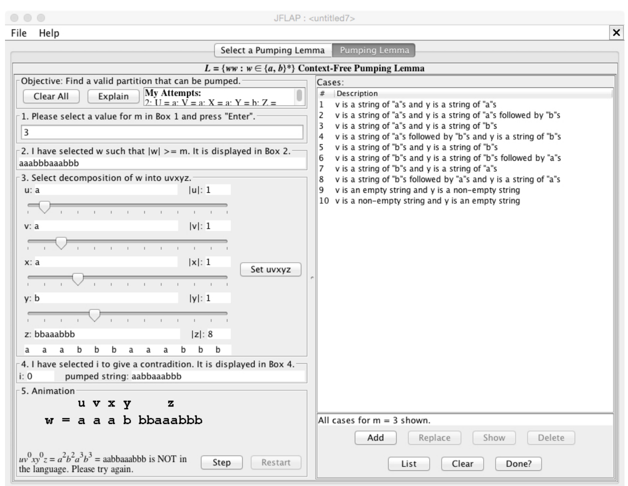

Figure 2: Step 2

We now use the sliders to choose appropriate strings u, v, x and y. When these choices are appropriate, you can click "Set $uvxyx$". In Figure 2, we see that $v = a$ and $y = b$, corresponding to Case 3 in the list. We see that JFLAP chooses $i = 0$ to arrive at a word uxz which is not in L, a contradiction. Repeat these steps for all other cases.

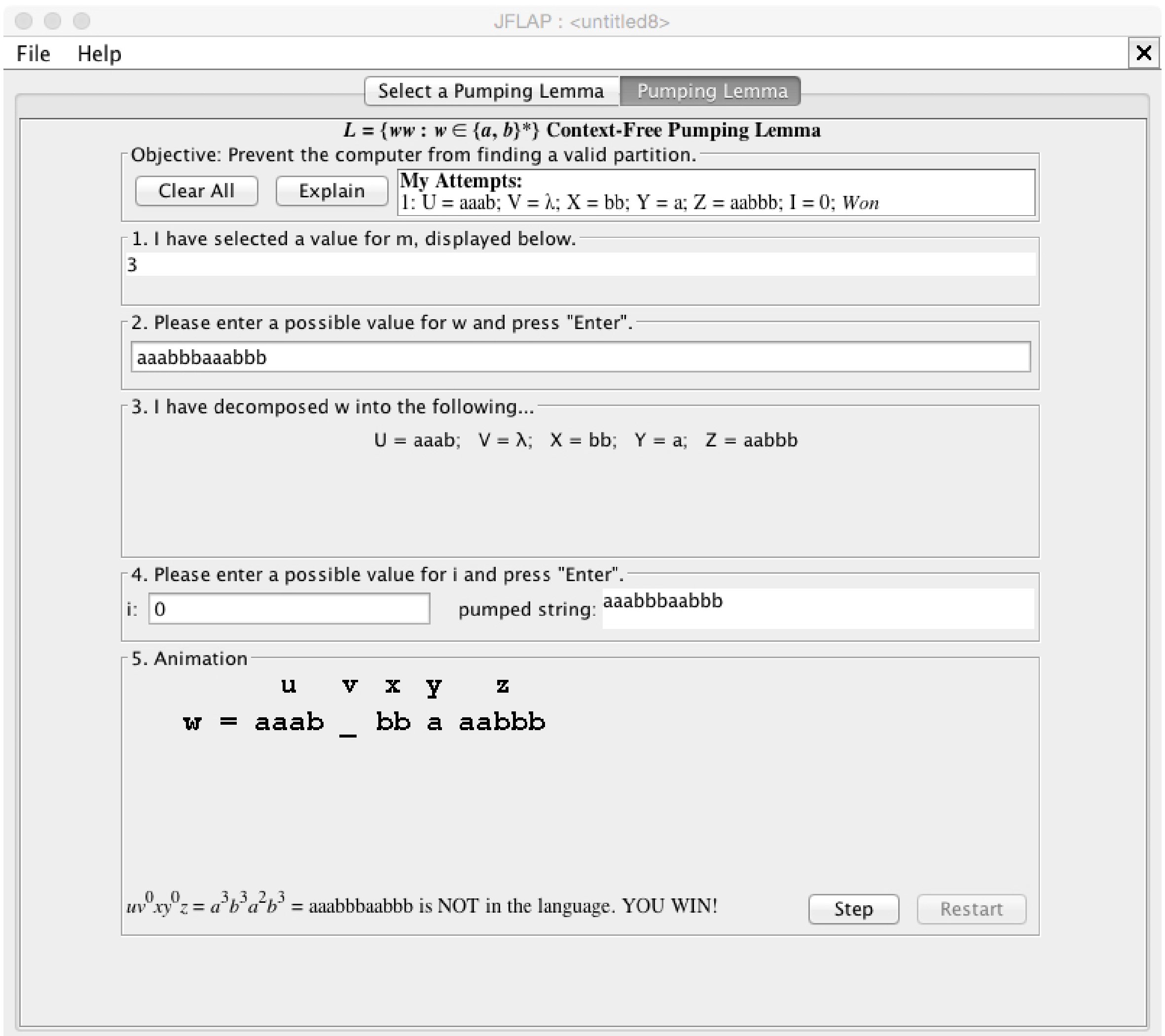

Figure 3: Computer goes first

We now do this again but this time we'll let the "Computer go first." Suppose it selects $m = 3$ as in Figure 3. Enter *aaabbbaaabbb* for w. JFLAP decomposes w. Select $i = 0$ in this case. You WIN. Repeat this entire process two or three times.

4 References

1. Introduction to the Theory of Computation (Third Edition), Michael Sipser. Cengage Learning. 2013.
2. JFLAP - An Interactive Formal Languages and Automata Package, Susan H. Rodger and Thomas W Finley. Jones and Bartlett Publishers. 2006